The distribution of localization centers in some discrete random systems

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Abstract

As a supplement of our previous work [10], we consider the localized region of the random Schrödinger operators on $l^2(\mathbf{Z}^d)$ and study the point process composed of their eigenvalues and corresponding localization centers. For the Anderson model, we show that, this point process in the natural scaling limit converges in distribution to the Poisson process on the product space of energy and space. In other models with suitable Wegner-type bounds, we can at least show that any limiting point processes are infinitely divisible.

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1 Introduction

The typical model we consider is the so-called Anderson model given below.

$$(H_\omega\varphi)(x) = \sum_{|x-y|=1} \varphi(y) + \lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbf{Z}^d)$$

where $\lambda > 0$ is the coupling constant and $\{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ are the independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The following facts are well-known.

(1) (the spectrum of H) the spectrum of H_ω is deterministic almost surely

$$\sigma(H_\omega) = \Sigma := [-2d, 2d] + \lambda \text{ supp } d\nu, \quad a.s.$$

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where ν is the distribution of $V_\omega(0)$ [13].

(2)(Anderson localization) There is an open interval $I \subset \Sigma$ such that with probability one, the spectrum of H_ω on I is pure point with exponentially decaying eigenfunctions. I can be taken (i) $I = \Sigma$ if λ is large enough, (ii) on the extreme energies, (iii) on the band edges, and (iv) away from the spectrum of the free Laplacian if λ sufficiently small [7, 16, 1, 2].

Recently, some relations between the eigenvalues and the corresponding localization centers are derived[15]. It roughly implies,
(1) If $|E - E_0| \simeq L^{-d}$, the localization center $x(E)$ corresponding to the energy E satisfies $|x(E)| \geq L$. Hence the distribution of the localization centers are thin in space.
(2) If $|E - E'| \simeq L^{-2d}$, the localization centers $x(E), x(E')$ corresponding to the energies E, E' satisfies $|x(E) - x(E')| \geq L$. Hence the localization centers are repulsive if the energies get closer.

On the other hand, in [10], they study the "natural scaling limit" of the random measure in $\mathbf{R}^{d+1}$ (the product of energy and space) composed of the eigenvalues and eigenfunctions. The result there roughly implies that the distribution of them with eigenvalues in the order of L^{-d} from the reference energy E_0, and with eigenfunctions in the order of L from the origin, obey the Poisson law on $\mathbf{R}^{d+1}$. This work can also be regarded as an extension of the work by Minami [14] who showed that the point process on $\mathbf{R}$ composed of the eigenvalues of H in the finite volume approximation converges to the Poisson process on $\mathbf{R}$. To summarize, [15, 10] imply that the eigenfunctions whose energies are in the order of L^{-d} are non-repulsive while those in the order of L^{-2d} are repulsive.

The aim of this paper is to supplement [10] : (i) to study the distribution of the localization centers which is technically different from what is done in [10] and (ii) to study what can be said to those models in which the Minami's estimate and the fractional moment bound, which are the main tool in [10], are currently not known to hold.

We set some notations.

Notation :
(1) For $x = (x_1, x_2, \cdots, x_d) \in \mathbf{Z}^d$, let $|x| = \sum_{j=1}^d |x_j|$. $\Lambda_L(x) := \{y \in \mathbf{Z}^d : |x - y| \leq \frac{L}{2}\}$ is the finite box in $\mathbf{Z}^d$ with length L centered at $x \in \mathbf{Z}^d$.

$|\Lambda| := \sharp\Lambda$ is the number of sites in the box Λ and χ_Λ is the characteristic function of Λ.
(2) For a box Λ, let

$$\tilde{\partial}\Lambda := \{\langle y, y'\rangle \in \Lambda \times \Lambda^c : |y - y'| = 1\}$$
$$\partial\Lambda := \left\{ y \in \Lambda : \langle y, y'\rangle \in \tilde{\partial}\Lambda \text{ for some } y' \in \Lambda^c \right\}$$

be two notions of the boundary of Λ.
(3) For a box $\Lambda(\subset \mathbf{Z}^d)$, $H_\Lambda := H|_\Lambda$ is the restriction of H on Λ. We adopt the zero(Dirichlet) boundary condition unless stated otherwise : $H_\Lambda := \chi_\Lambda H \chi_\Lambda$. For $E \notin \sigma(H_\Lambda)$, $G_\Lambda(E; x, y) = \langle \delta_x, (H_\Lambda - E)^{-1}\delta_y\rangle_{l^2(\Lambda)}$ is the Green function of H_Λ. $\delta_x \in l^2(\mathbf{Z}^d)$ is defined by $\delta_x(y) = 1(y = x), = 0(y \neq x)$ and $\langle \cdot, \cdot \rangle_{l^2(\Lambda)}$ is the inner-product on $l^2(\Lambda)$.
(4) Let $\gamma > 0, E \in \mathbf{R}$. We say that the box $\Lambda_L(x)$ is (γ, E)-regular iff $E \notin \sigma(H_{\Lambda_L(x)})$ and the following estimate holds [1]

$$\sup_{\epsilon > 0} |G_{\Lambda_L(x)}(E + i\epsilon; x, y)| \leq e^{-\gamma \frac{L}{2}}, \quad \forall y \in \partial\Lambda_L(x).$$

(5) For $\phi \in l^2(\mathbf{Z}^d)$, we define the set $X(\phi)$ of its localization centers by

$$X(\phi) := \left\{ x \in \mathbf{Z}^d : |\phi(x)| = \max_{y \in \mathbf{Z}^d} |\phi(y)| \right\}$$

This definition is due to [5]. Since $\phi \in l^2(\mathbf{Z}^d)$, $X(\phi)$ is a finite set. To be free from ambiguities, we choose $x(\phi) \in X(\phi)$ according to a certain order on $\mathbf{Z}^d$. For a box Λ, we say ϕ is localized in Λ iff $x(\phi) \in \Lambda$. If $\{E_j\}_j$, $\{\phi_j\}_j$ are the enumerations of the eigenvalues and eigenfunctions of H counting multiplicities, we set $X(E_j) := X(\phi_j), x(E_j) := x(\phi_j)$ and we say E_j is localized in Λ iff $x(E_j) \in \Lambda$. If an eigenvalue is degerated, we adopt any but fixed selection procedure of choosing eigenfunctions.
(6) For a Hamiltonian H, an interval $J(\subset \mathbf{R})$, and a box $\Lambda(\subset \mathbf{Z}^d)$, we set

$$N(H, J) := \sharp\{ \text{ eigenvalues of } H \text{ in } J \}$$
$$N(H, J, \Lambda) := \sharp\{ \text{ eigenvalues of } H \text{ in } J \text{ localized in } \Lambda \}$$

(7) For a n-dimensional measurable set $A(\subset \mathbf{R}^n)$, we denote by $|A|$ its Lebesgue measure. For $a \in \mathbf{R}$ and $r > 0$, $I(a, r) := \{x \in \mathbf{R} : |x - a| < r\}$ is

[1] We adopt this definition to treat Lemma 3.1 and Proposition 5.1.

the open interval centered at a with radius r.
(8) Set $K = [0,1]^d$ and let π_e and π_s be the canonical projections on $\mathbf{R} \times K$ onto $\mathbf{R}$ and K respectively : $\pi_e(E,x) = E$, $\pi_s(E,x) = x$ for $(E,x) \in \mathbf{R} \times K$.
(9) We set

$$\xi(f) = \int_{\mathbf{R}\times K} f(x)\xi(dx)$$

for a point process ξ and a bounded measurable function f on $\mathbf{R} \times K$.

We consider the following two assumptions.

Assumption A
(1) (Initial length scale estimate) Let $I(\subset \Sigma)$ be an open interval where the initial length scale estimate of the multiscale analysis holds : we can find $\gamma > 0$ and $p > 6d$ such that for sufficiently large L_0 we have

$$\mathbf{P}\left(\text{ For any } E \in I, \Lambda_{L_0}(0) \text{ is } (\gamma, E)\text{-regular }\right) \geq 1 - L_0^{-p}.$$

(2)(Wegner's estimate) We can find a positive constant C_W such that for any interval $J(\subset I)$ and any box Λ,

$$\mathbf{E}[N(H_\Lambda, J)] \leq C_W |\Lambda||J|.$$

Assumption A is known to hold, for instance, (1) for the Anderson model when the distribution of the random potential ν has the bounded density ρ, with the allowed location of I mentioned at the beginning of this section, (2) the Schrödinger operators with off-diagonal disorder [6], and (3) for the Schrödinger operators on $l^2(\mathbf{Z}^2)$ with random magnetic fluxes[12] (in (2), (3), I can be taken on the edge of the spectrum).
We need $p > 6d$ to eliminate the contributions from the negligible events, in the proof of Proposition 2.1.

Pick α with $1 < \alpha < \alpha_0 := \frac{2p}{p+2d} (< 2)$, and set

$$L_{k+1} = L_k^\alpha, \quad k = 0, 1, \cdots$$

For simplicity, we write $\Lambda_k(x) = \Lambda_{L_k}(x)$. By the multiscale analysis [16], we have, for $k = 1, 2, \cdots$ and for $x, y \in \mathbf{Z}^d$ with $|x - y| > L_k$,

$$\mathbf{P}\left(\text{ For any } E \in I, \text{ either } \Lambda_k(x) \text{ or } \Lambda_k(y) \text{ are } (\gamma, E)\text{-regular }\right) \geq 1 - L_k^{-2p}. \tag{1.1}$$

Assumption B (Minami's estimate)
We can find a positive constant C_M such that for any finite box Λ and any interval $J(\subset I)$,

$$\sum_{k=2}^{\infty} k(k-1)\mathbf{P}\left(N(H_\Lambda, J) = k\right) \leq C_M |\Lambda|^2 |J|^2.$$

Assumption B is known to be true for the Anderson model and for any interval $J(\subset \mathbf{R})$ when the potential distribution has the bounded density[14].

The integrated density of states $N(E)$ of H is defined by

$$N(E) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} N(H_\Lambda, (-\infty, E]).$$

It is known that, with probability one, this limit exists for any $E \in \mathbf{R}$ and continuous [3] so that its derivative $n(E)$ finitely exists a.e. $n(E)$ is called the density of states.

Let $M_p(\mathbf{R}^n)$ be the set of integer-valued Radon measures on $\mathbf{R}^n$ which is regarded as a metric space under the vague topology. The point process on $\mathbf{R}^n$ is defined to be the measurable mapping from $(\Omega, \mathcal{F}, \mathbf{P})$ to $M_p(\mathbf{R}^n)$. We say that a sequence $\{\xi_k\}$ of point process converges in distribution to a point process ξ and write $\xi_k \xrightarrow{d} \xi$ iff the distribution of ξ_k converges weakly to that of ξ. We formulate our problem below.

The formulation of the problem : Let $H_k = H|_{\Lambda_k}$ be the restriction of H on $\Lambda_k = \{1, 2, \cdots, L_k\}^d$ with the periodic boundary condition. The choice of this particular boundary condition is to be free from the boundary effect which should be purely technical. Let $E_1(\Lambda_k) \leq E_2(\Lambda_k) \leq \cdots \leq E_{|\Lambda_k|}(\Lambda_k)$ be the eigenvalues of H_k in increasing order and let $x_j = x(E_j(\Lambda_k)) \in X(E_j(\Lambda_k))$ be the corresponding localization center. Take a reference energy $E_0 \in I$ and define the point process on $\mathbf{R} \times K$ as follows.

$$\xi_k = \sum_{j=1}^{|\Lambda_k|} \delta_{X_j}, \quad X_j = (|\Lambda_k|(E_j(\Lambda_k) - E_0), L_k^{-1} x_j) \in \mathbf{R} \times K, \quad K = [0,1]^d.$$

This scaling is the same as that in [14, 10] : the energies are supposed to accumulate in the order of L^{-d} around E_0 for large L if $n(E_0) < \infty$, and if $|E - E_0| \simeq L^{-d}$, we expect $|x(E)| \simeq L$ [15, Theorem 1.1]. The main theorem of this paper is

Theorem 1.1 *Assume Assumptions A, B. If $n(E_0) < \infty$, then $\xi_k \xrightarrow{d} \zeta_{P,\mathbf{R}\times K}$ as $k \to \infty$ where $\zeta_{P,\mathbf{R}\times K}$ is the Poisson process on $\mathbf{R} \times K$ with its intensity measure $n(E_0)dE \times dx$.*

If we only assume Assumption A, we can find convergent subsequence and its limiting point process is infinitely divisible whose intensity measure is absolutely continuous(Theorem 2.1). (we say the point process ξ is infinitely divisible iff for any $n \in \mathbf{N}$, we can find i.i.d. array of point process $\{\xi_{nj}\}_{j=1}^{n}$ with $\xi \overset{d}{=} \sum_{j=1}^{n} \xi_{nj}$). The same conclusion is proved in [8] for the one-dimensional Schrödinger operator on $\mathbf{R}$. The infinite divisibility of ξ merely implies that ξ is represented as the Poisson process on $M_p(\mathbf{R} \times K)$ whose intensity measure is given by the canonical measure of ξ[9, Lemma 6.5, 6.6]. We are unable to prove Theorem 1.1 if we replace H_k by H itself (which is done in [10]) for some "a priori" estimates are missing to prove Step 1 in Proposition 2.1, Lemma 4.4 and Lemma 4.7.

By "projecting" the result of Theorem 1.1 to the energy and space axis respectively, we have the following facts.

(1) (Absence of Repulsion)
Let $\xi(\Lambda, E_0)$ be the point process composed of the eigenvalues $\{E_j(\Lambda)\}_{j=1}^{|\Lambda|}$ of H_Λ :

$$\xi(\Lambda, E_0) := \sum_j \delta_{|\Lambda|(E_j(\Lambda)-E_0)}.$$

Then we recover the result in [14].

Corollary 1.1 *Under the same condition as Theorem 1.1, we have $\xi(\Lambda_k, E_0) \xrightarrow{d} \zeta_{P,\mathbf{R}}$ as $k \to \infty$ where $\zeta_{P,\mathbf{R}}$ is the Poisson process on $\mathbf{R}$ with intensity measure $n(E_0)dE$.*

Because we assumed one of the essential ingredients of the proof [14, Lemma 2] as Assumption B, it is not an alternative proof.

(2) (Distribution of the localization centers)
For an interval $J(\subset \mathbf{R})$, let $\{F_j(\Lambda_k, J)\}_{j\geq 1}$ be the eigenvalues of H_k in $(E_0 + L_k^{-d} J)$ and choose $x_j(\Lambda_k) \in X(F_j(\Lambda_k, J))$. Define the point process on K by

$$\xi_{k,loc} = \xi_k(J \times \cdot) = \sum_{j\geq 1} \delta_{L_k^{-1} x_j(\Lambda_k)}$$

Corollary 1.2 *Under the same assumption as in Theorem 1.1,* $\xi_{k,loc} \xrightarrow{d} \zeta_{P,K}$ *as* $k \to \infty$ *where* $\zeta_{P,K}$ *is the Poisson process on* K *with intensity measure* $n(E_0)|J|dx$.

The remaining sections are organized as follows. In Section 2, we prove Theorem 2.1 which is one of the main steps to apply the Poisson convergence theorem [9, Corollary 7.5] to prove Theorem 1.1. In order to do that, we decompose Λ_k into disjoint boxes $\{D_p\}_p$ of size L_{k-1}, and let $H_p = H|_{D_p}$ as is done in [14]. Since the eigenfunctions of H_k corresponding to the eigenvalue E in I are exponentially localized, we can find a box D_p such that H_p has eigenvalues near E. By some perturbative argument, we can construct a one to one correspondence between the eigenvalues of H_k and that of $\oplus_p H_p$, with probability close to 1. Therefore, ξ_k is approximated by the sum $\eta_k = \sum_p \eta_{k,p}$ of the point process composed of the eigenvalues and localization centers of H_p. Wegner's estimate ensures that $\{\eta_{k,p}\}_{k,p}$ is a null-array and relatively compact, so that $\{\eta_k\}_k$ always has the convergent subsequence whose limiting point is infinitely divisible.

In Section 3, under Assumption A, B, we show that η_k converges in distribution to the Poisson process. By Minami's estimate, $\eta_{k,p}$ has at most one atom in the corresponding region in $\mathbf{R} \times K$ with the probability close to 1. Hence the general Poisson convergence theorem [9, Corollary 7.5] gives the result. Since the mechanism to converge to the Poisson process is the same as in [14], Theorem 1.1 can be regarded as the extension of that.

To construct that one to one correspondence, we used the machinery developed in [5, 11] which is reviewed in Section 4.

For the random measure studied in [10], we can show the same statement as in Theorem 2.1 under Assumption A, which is mentioned in Section 5 with some remarks.

If we also assume Assumption B in the proof of Proposition 2.1, H_p has at most one eigenvalues in the corresponding region with probability close to 1, so that the correspondence between eigenvalues of H_{k+1} and H_p becomes bijective apart from negligible contributions, which is mentioned in Section 6.

2 Infinite Divisibility

For simplicity, we consider ξ_{k+1} instead of ξ_k. We first decompose Λ_{k+1} into disjoint cubes D_p of size L_k : $\Lambda_{k+1} = \bigcup_{p=1}^{N_k} D_p$, $N_k = \left(\frac{L_{k+1}}{L_k}\right)^d (1+o(1))$. The contributions of boxes near the boundary of Λ_{k+1} turn out to be negligible by Lemma 2.1. We denote by C_p the box which has the same center as D_p of size $L_k - 2L_{k-1}$: C_p is obtained by eliminating the strip of width L_{k-1} from the boundary of D_p.

$$C_p := \{x \in D_p : d(x, \partial D_p) \ge L_{k-1}\}.$$

Let $H_{k,p} := H|_{D_p}$ with the periodic boundary condition. We set the following event

$$\Omega_k = \Big\{ \omega \in \Omega : \text{For any} E \in I, \text{ either } \Lambda_{k-1}(x) \text{ or } \Lambda_{k-1}(y) \text{ are } (\gamma, E)\text{-regular} \\ \text{for any } x, y \in \Lambda_{k+1} \text{ with } |x-y| > L_{k-1} \Big\}$$

which by (1.1) satisfies

$$\mathbf{P}(\Omega_k) \ge 1 - L_{k-1}^{-2p} L_{k+1}^{2d} = 1 - L_{k-1}^{-2p+2d\alpha^2}. \tag{2.1}$$

We define the point process by

$$\eta_{k+1} = \sum_{p=1}^{N_k} \eta_{k+1,p}, \quad \eta_{k+1,p} = \sum_{j=1}^{|D_p|} \delta_{Y_{p,j}}$$
$$Y_{p,j} = (|\Lambda_{k+1}|(E_j(D_p) - E_0), L_{k+1}^{-1} y_{p,j})$$

where $E_1(D_p) \le E_2(D_p) \le \cdots \le E_{|D_p|}(D_p)$ are the eigenvalues of $H_{k,p}$, with $y_{p,j} = x(E_j(D_p)) \in X(E_j(D_p))$ their corresponding localization centers. As was explained in Introduction, we expect that ξ_{k+1} can be approximated by $\sum_p \eta_{k+1,p}$ to be shown below.

Proposition 2.1 *Under Assumption A, we have*

$$\mathbf{E}\left[|\xi_{k+1}(f) - \eta_{k+1}(f)|\right] \to 0, \quad k \to \infty, \quad f \in C_c(\mathbf{R} \times K).$$

Remark 2.1 *By Proposition 2.1, the Laplace transform* $L_\xi(f) = \mathbf{E}[e^{-\xi(f)}]$ *of* ξ *satisfies*

$$L_{\xi_{k+1}}(f) - L_{\eta_{k+1}}(f) \to 0, \quad k \to \infty, \quad f \in C_c^+(\mathbf{R} \times K).$$

Hence it suffices to show

$$\eta_{k+1} \xrightarrow{d} \zeta_{P,\mathbf{R}\times K}$$

to prove Theorem 1.1.

Remark 2.2 *By choosing* f *independent of the space variables, we obtain an alternative proof of [14, Step 3]. Since we use the exponential decay of eigenfunctions instead of that of Green's function, this proof is mathematically indirect but physically direct.*

Proof. Step 1 : We show the contribution by the event Ω_k^c is negligible. In fact, since $|\xi_{k+1}(f)| \le \|f\|_\infty |\Lambda_{k+1}|$ and since $p > 6d > \frac{3}{2}d\alpha^2$, we have

$$\mathbf{E}[|\xi_{k+1}(f)|; \Omega_k^c] \le \|f\|_\infty |\Lambda_{k+1}| L_{k-1}^{-2p+2d\alpha^2} = (const.) L_{k-1}^{-2p+3d\alpha^2} = o(1)$$

by (2.1)[2]. $\mathbf{E}[\sum_p \eta_{k+1,p}(f); \Omega_k^c]$ can be estimated similarly. Therefore, it suffices to show

$$\mathbf{E}\left[|\xi_{k+1}(f) - \eta_{k+1}(f)|; \Omega_k\right] = o(1).$$

Step 2 : We show the contribution by the atoms whose localization centers are in $\bigcup_p (D_p \setminus C_p)$ are negligible. We first decompose

$$\begin{aligned}
\xi_{k+1} &= \xi_{k+1}^{(1)} + \xi_{k+1}^{(2)}, \\
\xi_{k+1}^{(j)} &= \sum_{p=1}^{N_k} \xi_{k+1,p}^{(j)}, \quad j = 1, 2, \\
\xi_{k+1,p}^{(1)} &= \sum_{x_j \in C_p} \delta_{X_j}, \quad \xi_{k+1,p}^{(2)} = \sum_{x_j \in D_p \setminus C_p} \delta_{X_j}.
\end{aligned}$$

And we decompose $\eta_{k+1,p}$ similarly. In what follows, we take any $0 < \gamma' < \gamma$ and let k large enough with $k \ge k_2(\alpha, d, \gamma, \gamma') \vee k_3(\alpha, d, \gamma, \gamma')$ and $k \ge k_2'(\alpha, d, \gamma, \gamma') \vee k_3'(\alpha, d, \gamma, \gamma')$, where k_2, k_3, k_2' and k_3' are defined in Lemmas 4.3, 4.4, 4.6 and 4.7 respectively. For simplicity, set

$$\epsilon_{k-1} := e^{-\gamma' L_{k-1}/2}.$$

[2] the equation "$\cdots = o(1)$" henceforth means "$\cdots = o(1)$ as $k \to \infty$".

Claim 1

$$\mathbf{E}[\xi^{(2)}_{k+1}(f);\Omega_k] = o(1), \quad \mathbf{E}[\eta^{(2)}_{k+1}(f);\Omega_k] = o(1).$$

Proof of Claim 1 Let

$$S_p = \{x \in \Lambda_{k+1} : d(x, \partial(D_p \setminus C_p)) \le L_{k-1}\}, \quad H'_{k,p} := H_{k+1}|_{S_p}$$

(with the Dirichlet boundary condition). Pick $a > 0$ with $\pi_e(\text{ supp } f) \subset [-a, a]$ and set

$$J_{k+1} = \left[E_0 - \frac{a}{|\Lambda_{k+1}|}, E_0 + \frac{a}{|\Lambda_{k+1}|}\right] = I\left(E_0, \frac{a}{|\Lambda_{k+1}|}\right).$$

For $\omega \in \Omega_k$, we have

$$N(H_{k+1}, J_{k+1}, D_p \setminus C_p) \le N(H'_{k,p}, J_{k+1} + I(0, \epsilon_{k-1})),$$

by Lemma 4.4(2). By Assumption A(2), we have

$$\begin{aligned}
\mathbf{E}[\xi^{(2)}_{k+1,p}(f);\Omega_k] &\le \|f\|_\infty \mathbf{E}[N(H_{k+1}, J_{k+1}, D_p \setminus C_p)] \\
&\le \|f\|_\infty \mathbf{E}[N(H'_{k,p}, J_{k+1} + I(0, \epsilon_{k-1}))] \\
&\le (const.)\|f\|_\infty C_W |D_p \setminus C_p| \cdot \frac{2a}{|\Lambda_{k+1}|}.
\end{aligned}$$

Using the inequality $|D_p \setminus C_p| \le (const.) L_{k-1} L_k^{d-1}$ and then taking sum w.r.t. p gives

$$\mathbf{E}[\xi^{(2)}_{k+1}(f);\Omega_k] \le (const.)\frac{L_{k-1}}{L_k} = o(1).$$

To estimate $\eta^{(2)}_{k+1}$, we set

$$T_p = \{x \in D_p : d(x, \partial(D_p \setminus C_p)) \le L_{k-1}\}, \quad H''_{k,p} := H_{k,p}|_{T_p}.$$

Then the same argument as above with Lemma 4.4(3) gives $\mathbf{E}[\eta^{(2)}_{k+1}(f);\Omega_k] \le (const.)\frac{L_{k-1}}{L_k} = o(1)$ and thus proves Claim 1. □

Therefore, it suffices to show

$$\mathbf{E}\left[|\xi^{(1)}_{k+1}(f) - \eta_{k+1}(f)|;\Omega_k\right] = o(1).$$

The equation $\mathbf{E}[\eta^{(2)}_{k+1}(f);\Omega_k]=o(1)$ will be used in Step 3 below.

Step 3 : We first show the following claim.

Claim 2 Let $J\subset I$ be an interval. If $\omega\in\Omega_k$, we have
(1)
$$N(H_{k+1},J,C_p)\le N(H_{k,p},J+I(0,\epsilon_{k-1})),$$
(2)
$$\begin{aligned}&\sum_p N(H_{k,p},J+I(0,\epsilon_{k-1}))\\ &\le \sum_p N(H_{k+1},J,C_p)+\sum_p N(H_{k+1},(J+I(0,2\epsilon_{k-1}))\setminus J,C_p)\\ &\qquad+\sum_p N(H'_{k,p},J+I(0,3\epsilon_{k-1}))+\sum_p N(H''_{k,p},J+I(0,2\epsilon_{k-1})).\end{aligned}$$

Proof of Claim 2 (1) clearly follows from Lemma 4.4. To show (2), we decompose

$$\begin{aligned}N(H_{k,p},J+I(0,\epsilon_{k-1})) &= N(H_{k,p},J+I(0,\epsilon_{k-1}),C_p)+N(H_{k,p},J+I(0,\epsilon_{k-1}),D_p\setminus C_p)\\ &=: I_p+II_p.\end{aligned}$$

By Lemma 4.4,
$$II_p\le N(H''_{k,p},J+I(0,2\epsilon_{k-1}))$$
and by Lemma 4.4 and Lemma 4.7,

$$\begin{aligned}\sum_p I_p &\le N(H_{k+1},J+I(0,2\epsilon_{k-1}))\\ &=\sum_p N(H_{k+1},J+I(0,2\epsilon_{k-1}),C_p)+\sum_p N(H_{k+1},J+I(0,2\epsilon_{k-1}),D_p\setminus C_p)\\ &\le\sum_p N(H_{k+1},J+I(0,2\epsilon_{k-1}),C_p)+\sum_p N(H'_{k,p},J+I(0,3\epsilon_{k-1}))\end{aligned}$$

which shows Claim 2(2). □

For any $p=1,2,\cdots,N_k$, let $\{E_{p,j}\}$ be the eigenvalues of H_{k+1} in J_{k+1} localized in C_p, and write $\bigcup_j I(E_{p,j},\epsilon_{k-1})$ as the disjoint union of open intervals
:
$$\bigcup_j I(E_{p,j},\epsilon_{k-1})=\bigcup_i I_i.$$

If $a_1^{(i)} < a_2^{(i)} < \cdots < a_{N_i}^{(i)}$ are the eigenvalues of H_{k+1} in I_i localized in C_p, then

$$I_i = I_i' + I(0, \epsilon_{k-1}), \quad I_i' := (a_1^{(i)}, a_{N_i}^{(i)}).$$

Letting $J = I_i'$ in Claim 2(1), we have

$$N(H_{k+1}, I_i, C_p) \leq N(H_{k,p}, I_i)$$

and hence we have an one to one correspondence from the eigenvalues of H_{k+1} in I_i localized in C_p to those of $H_{k,p}$ in I_i. Since diam $(I_i) \leq L_k^d \epsilon_{k-1}$, the corresponding eigenvalues $E_{j,p}, F_{j,p}$ of $H_{k+1}, H_{k,p}$ satisfy

$$|E_{j,p} - F_{j,p}| \leq L_k^d \epsilon_{k-1}.$$

On the other hand, by letting $J = J_{k+1}$ in Claim (2) we see that, the number of eigenvalues of $H_{k,p}$ in $J_{k+1} + I(0, \epsilon_{k-1})$ for $p = 1, 2, \cdots, N_k$ which do not lie in the range of this correspondence is less than

$$\sum_p N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1}, C_p) + \sum_p N(H'_{k,p}, J_{k+1} + I(0, 3\epsilon_{k-1}))$$
$$+ \sum_p N(H''_{k,p}, J_{k+1} + I(0, 2\epsilon_{k-1})).$$

Therefore, if $x_{j,p} = x(E_{j,p})$ (resp. $y_{j,p} = x(F_{j,p})$) are the localization center of $E_{j,p}$ (resp. $F_{j,p}$), we have

$$\begin{aligned}
&\mathbf{E}\left[\sum_p \left|\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)\right| ; \Omega_k\right] \\
\leq\ &\mathbf{E}\left[\sum_p \sum_j \left| f(|\Lambda_{k+1}|(E_{j,p} - E_0), L_{k+1}^{-1} x_{j,p}) - f(|\Lambda_{k+1}|(F_{j,p} - E_0), L_{k+1}^{-1} y_{j,p})\right| ; \Omega_k\right] \\
&+ \sum_p \|f\|_\infty \mathbf{E}\left[N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1}, C_p)\right] \\
&+ \sum_p \|f\|_\infty \mathbf{E}\left[N(H'_{k,p}, J_{k+1} + I(0, 3\epsilon_{k-1}))\right] \\
&+ \sum_p \|f\|_\infty \mathbf{E}\left[N(H''_{k,p}, J_{k+1} + I(0, 2\epsilon_{k-1}))\right] \\
=:\ &I + II + III + IV.
\end{aligned}$$

Since f is uniformly continuous, for any $\epsilon > 0$ we have $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$ with some $\delta(\epsilon) > 0$. Since

$$\begin{aligned}
&\left|\Big(|\Lambda_{k+1}|(E_{j,p} - E_0), L_{k+1}^{-1} x_{j,p}\Big) - \Big(|\Lambda_{k+1}|(F_{j,p} - E_0), L_{k+1}^{-1} y_{j,p}\Big)\right| \\
&\le |\Lambda_{k+1}| L_{k+1}^d \epsilon_{k-1} + L_k / L_{k+1} < \delta(\epsilon)
\end{aligned}$$

for large k, we have

$$I \le \epsilon\, \mathbf{E}\left[\sum_p N(H_{k+1}, J_{k+1}, C_p)\right] \le \epsilon\, C_W \frac{2a}{|\Lambda_{k+1}|} |\Lambda_{k+1}| = (const.)\, \epsilon$$

by Assumption A(2), which also gives a bound for II.

$$\begin{aligned}
II &\le \|f\|_\infty \mathbf{E}\left[N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1})\right] \\
&\le \|f\|_\infty C_W 4 e^{-\gamma' L_{k-1}/2} |\Lambda_{k+1}| = o(1).
\end{aligned}$$

III, IV can be estimated similarly as in Step 2 :

$$\begin{aligned}
III,\, IV &\le \sum_p \|f\|_\infty C_W |D_p \setminus C_p| \left(\frac{2a}{|\Lambda_{k+1}|} + 6\epsilon_{k-1}\right) \\
&\le (const.) \|f\|_\infty \left(\frac{L_{k+1}}{L_k}\right)^d C_W L_{k-1} L_k^{d-1} L_{k+1}^{-d} \\
&\le (const.) \frac{L_{k-1}}{L_k}.
\end{aligned}$$

The proof of Proposition 2.1 is now completed. $\square$

We next show some elementary bounds of ξ_{k+1}, η_{k+1} to study their basic properties.

Lemma 2.1
(1) For a bounded interval $A (\subset \mathbf{R} \times K)$,

$$\mathbf{E}[\xi_{k+1}(A)] \le C_W |\pi_e(A)|, \quad \mathbf{E}[\eta_{k+1,p}(A)] \le C_W \frac{|\pi_e(A)|}{|\Lambda_{k+1}|} \cdot |\Lambda_k|$$

for large k.
(2) For $f \in C_c(\mathbf{R} \times K)$ we have

$$\mathbf{E}[\sum_p |\eta_{k+1,p}(f)|] \le (const.) C_W \|f\|_1$$

for large k.

Proof. (1) Since $J_{k+1} := E_0 + |\Lambda_{k+1}|^{-1}\pi_e(A) \subset I$ for k large enough, Assumption A(2) gives

$$\mathbf{E}[\xi_{k+1}(A)] \le \mathbf{E}[N(H_{k+1}, J_{k+1})] \le C_W|\pi_e(A)|$$
$$\mathbf{E}[\eta_{k+1,p}(A)] \le \mathbf{E}[N(H_{k,p}, J_{k+1})] \le C_W \frac{|\pi_e(A)|}{|\Lambda_{k+1}|} \cdot |\Lambda_k|.$$

(2) Let $A = J \times B$ ($J \subset \mathbf{R}, B \subset K$) be an interval. Taking $D'_p = L_{k+1}^{-1} D_p$ we have

$$\sum_p \eta_{k+1,p}(A) \le \sum_{p: D'_p \cap B \neq \emptyset} N(H_{k,p}, E_0 + L_{k+1}^{-d} J).$$

Since

$$\sharp \left\{ p : D'_p \cap B \neq \emptyset \right\} \le (const.) \frac{(L_{k+1}B)^d}{L_k^d} \le (const.) \frac{|B| \cdot |\Lambda_{k+1}|}{|\Lambda_k|},$$

we obtain, using Assumption A(2),

$$\mathbf{E}[\sum_p \eta_{k+1,p}(A)] \le (const.) \frac{|B| \cdot |\Lambda_{k+1}|}{|\Lambda_k|} \cdot C_W \frac{|J|}{|\Lambda_{k+1}|} |\Lambda_k| = (const.) C_W |A|.$$

A density argument gives the result. □

The following lemma easily follows from Lemma 2.1(1).

Lemma 2.2
(1) $\{\eta_{k+1,p}\}_{p=1}^{N_k}$ is a null-array, i.e., for any bounded interval $A(\subset \mathbf{R} \times K)$,

$$\lim_{k\to\infty} \sup_{1 \le p \le N_k} \mathbf{P}(\eta_{k+1,p}(A) \ge 1) = 0.$$

(2) We have the following equation

$$\lim_{t\to\infty} \limsup_{k\to\infty} \mathbf{P}\left(\sum_p \eta_{k+1,p}(A) \ge t \right) = 0.$$

Hence by [9, Lemma 4.5], $\{\sum_p \eta_{k+1,p}\}_k$ is relatively compact.

We sum up the results obtained in this section.

Theorem 2.1 *Assume Assumption A and $n(E_0) < \infty$. Then $\{\xi_k\}$ has a convergent subsequence and the limiting point ξ is infinitely divisible whose intensity measure satisfies*

$$\mathbf{E}[\xi(A)] \leq n(E_0)|A|, \quad A \in \mathcal{B}(\mathbf{R} \times K).$$

Proof. The infinite divisibility follows from [9, Theorem 6.1], Proposition 2.1 and Lemma 2.2. The claim for the intensity measure follows from the following three considerations.
(1) If $\xi_k \xrightarrow{d} \xi$, then $\xi_k f \xrightarrow{d} \xi f$ for $f \in C_c(\mathbf{R} \times K), f \geq 0$ [9, Lemma 4.4]. Hence

$$\mathbf{E}[\xi(f)] \leq \liminf_{k \to \infty} \mathbf{E}[\xi_{k+1}(f)].$$

(2) By a density argument using Lemma 2.1(2), we deduce from (3.4) (note that Assumption B is not used to derive (3.4))

$$\mathbf{E}[\sum_p \eta_{k+1,p}(f)] \to n(E_0)\|f\|_1, \quad f \in C_c(\mathbf{R} \times K).$$

(3) By Proposition 2.1,

$$\mathbf{E}[\sum_p \eta_{k+1,p}(f)] - \mathbf{E}[\xi_{k+1}(f)] \to 0, \quad f \in C_c(\mathbf{R} \times K).$$

□

3 Poisson Limit Theorem

In this section, we show that $\{\xi_k\}$ converges in distribution to the Poisson process, under Assumption A, B. The two conditions in the following Proposition are sufficient to prove that.

Proposition 3.1 *Under Assumption A, B, we have, for a bounded interval $A(\subset \mathbf{R} \times K)$,*

$$(1) \quad \sum_p \mathbf{P}(\eta_{k+1,p}(A) \geq 2) \to 0, \tag{3.1}$$

$$(2) \quad \sum_p \mathbf{P}(\eta_{k+1,p}(A) \geq 1) \to n(E_0)|A|. \tag{3.2}$$

Proposition 3.1 together with [9, Corollary 7.5], Proposition 2.1 and Lemma 2.2 proves Theorem 1.1. For its proof, a preparation is necessary.

Lemma 3.1 *Assume Assumption A. For an interval $J(\subset \mathbf{R})$, we have*

$$\sum_{p=1}^{N_k} \mathbf{E}[\eta_{k+1,p}(J \times K)] \to n(E_0)|J|.$$

Proof. Since $\mathbf{E}[|\xi_{k+1}(J \times K) - \sum_p \eta_{k+1,p}(J \times K)|] \to 0$ by Proposition 2.1 and Lemma 2.1(1), it suffices to show

$$\mathbf{E}[\xi_{k+1}(J \times K)] \to n(E_0)|J|.$$

As is done in [14], it is further sufficient to show the above equality for the following function instead of 1_J

$$f_\zeta(x) = \frac{\tau}{(x-\sigma)^2 + \tau^2}, \quad \zeta = \sigma + i\tau \in \mathbf{C}_+,$$

because the set

$$\mathcal{A} := \left\{ \sum_{j=1}^{n} a_j f_{\zeta_j}(x) : \ a_j \geq 0, \ \zeta_j \in \mathbf{C}_+ \right\}$$

of the finite linear combinations of f_ζ with positive coefficients is dense in $L^1_+(\mathbf{R})$ [14, Lemma 1], and Lemma 4.8 enables us to carry out the density argument. Hence it suffices to show

$$\mathbf{E}[\xi_{k+1}(f_\zeta)] \to \pi n(E_0), \quad \zeta \in \mathbf{C}_+.$$

For any $x \in \Lambda_{k+1}$, we have

$$\begin{aligned} \mathbf{E}[\xi_{k+1}(f_\zeta)] &= \frac{1}{|\Lambda_{k+1}|}\mathbf{E}[\mathrm{Tr}\ \Im G_{k+1}(E_0 + \frac{\zeta}{|\Lambda_{k+1}|})] \\ &= \mathbf{E}[\Im G_{k+1}(E_0 + \frac{\zeta}{|\Lambda_{k+1}|}; x, x)] \end{aligned}$$

(since we impose the periodic boundary condition). Let $G(z) = (H-z)^{-1}$ be Green's function of H. Let x be the center of Λ_{k+1} and let $z_{k+1} = E_0 + \frac{\zeta}{|\Lambda_{k+1}|}$. Then by the resolvent equation,

$$\begin{aligned} &G_{k+1}(z_{k+1}; x, x) - G(z_{k+1}; x, x) \\ &\quad = \sum_{\langle y, y' \rangle \in \tilde{\partial}\Lambda_{k+1}} G_{k+1}(z_{k+1}; x, y) G(z_{k+1}; y', x) \end{aligned}$$

By the multiscale analysis, the event

$$\mathcal{G}_{k+1}(E) := \{\omega \in \Omega : \ \Lambda_{k+1} \text{ is } (\gamma_0, E)\text{-regular } \}$$

satisfies

$$\mathbf{P}\left(\mathcal{G}_{k+1}(E)\right) \geq 1 - L_{k+1}^{-p}$$

for any $E \in I$ and $0 < \gamma_0 < \gamma$. Take k large enough and let $E_{k+1} = \Re z_{k+1}$. We decompose

$$\begin{aligned}
&|\mathbf{E}[G_{k+1}(z_{k+1}; x, x) - \mathbf{E}[G(z_{k+1}; x, x)]]) \\
\leq& \sum_{\langle y, y' \rangle \in \partial\Lambda_{k+1}(x)} \mathbf{E}[|G_{k+1}(z_{k+1}; x, y)| \cdot |G(z_{k+1}; y', x)|; \mathcal{G}_{k+1}(E_{k+1})] \\
&\quad + \sum_{\langle y, y' \rangle \in \partial\Lambda_{k+1}(x)} \mathbf{E}[|G_{k+1}(z_{k+1}; x, y)| \cdot |G(z_{k+1}; y', x)|; \mathcal{G}_{k+1}(E_{k+1})^c] \\
=:& \ I + II.
\end{aligned}$$

Because

$$I \leq c_d L_{k+1}^{d-1} e^{-\gamma_0 \frac{L_{k+1}}{2}} L_{k+1}^d = o(1), \quad II \leq c_d L_{k+1}^{d-1} L_{k+1}^{2d} L_{k+1}^{-p},$$

we need $p > 3d - 1$ to have $II = o(1)$, which is guaranteed by Assumption A(1). Therefore

$$\begin{aligned}
\mathbf{E}[\xi_{k+1}(f_\zeta)] &= \frac{1}{|\Lambda_{k+1}|} \mathbf{E}[\mathrm{Tr}\ \Im G_{k+1}(z_{k+1})] \\
&= \mathbf{E}[\Im G(z_{k+1}; x, x)] + o(1) \\
&= \pi n(E_0) + o(1).
\end{aligned}$$

as $k \to \infty$. $\square$

Proof of Proposition 3.1 Let $A (\subset \mathbf{R} \times K)$ be a bounded interval. As is discussed in [14], it suffices to show the following equations to prove Proposition 3.1.

$$(1) \quad \sum_{j \geq 2} \sum_p \mathbf{P}(\eta_{k+1,p}(A) \geq j) \to 0, \tag{3.3}$$

$$(2) \quad \sum_p \mathbf{E}[\eta_{k+1,p}(A)] \to n(E_0)|A|. \tag{3.4}$$

In fact, (3.3) trivially implies (3.1), and (3.2) follows from

$$\sum_p \mathbf{P}(\eta_{k+1,p}(A) \geq 1) = \sum_p \mathbf{E}[\eta_{k+1,p}(A)] - \sum_p \sum_{j\geq 2} \mathbf{P}(\eta_{k+1,p}(A) \geq j) \to n(E_0)|A|.$$

(3.3) in turn follows from Assumption B :

$$\begin{aligned}
\sum_p \sum_{j\geq 2} \mathbf{P}(\eta_{k+1,p}(A) \geq j) &\leq \sum_p \sum_{j\geq 2} (j-1)\mathbf{P}(\eta_{k+1,p}(\pi_e(A)\times K) = j) \\
&\leq \sum_p \sum_{j\geq 2} j(j-1)\mathbf{P}(\eta_{k+1,p}(\pi_e(A)\times K) = j) \\
&\leq C_M N_k \frac{|\Lambda_k|^2}{|\Lambda_{k+1}|^2} \to 0
\end{aligned}$$

which is the only (and fundamental) step to use Assumption B.

To prove (3.4), let $J = \pi_e(A), B = \pi_s(A)$ and let $D'_p = L_{k+1}^{-1} D_p$. We then have

$$\begin{aligned}
\sum_p \mathbf{E}[\eta_{k+1,p}(A)] &= \sum_{B\cap D'_p \neq \emptyset} \mathbf{E}[\eta_{k+1,p}(A)] \\
&= \sum_{D'_p \subset B} \mathbf{E}[\eta_{k+1,p}(A)] + \sum_{B\cap D'_p\neq\emptyset, B\cap D'^c_p\neq\emptyset} \mathbf{E}[\eta_{k+1,p}(A)] \\
&=: I + II. \qquad (3.5)
\end{aligned}$$

By Lemma 2.1(1) and by the inequality

$$\sharp\{p : D'_p \cap \pi_s(A) \neq \emptyset, D'_p \cap \pi_s(A)^c \neq \emptyset\} \leq (const.) \left(\frac{L_{k+1}}{L_k}\right)^{d-1},$$

we have

$$II \leq (const.) \left(\frac{L_{k+1}}{L_k}\right)^{d-1} \cdot \frac{|\Lambda_k|}{|\Lambda_{k+1}|} = (const.) \frac{L_k}{L_{k+1}}. \qquad (3.6)$$

To compute I, we note

$$I = \sharp\{p : D'_p \subset B\} \mathbf{E}[\eta_{k+1,p}(J\times K)].$$

Substituting

$$N_k \mathbf{E}[\eta_{k+1,p}(J\times K)] = n(E_0)|J| + o(1),$$

which follows from Lemma 3.1, we have

$$I = \frac{\sharp\{p : D'_p \subset B\}}{N_k} \left(n(E_0)|J| + o(1)\right) = n(E_0)|B|\cdot|J| + o(1) \qquad (3.7)$$

as $k \to \infty$. By (3.5), (3.6) and (3.7), we obtain (3.4). $\square$

4 Appendix 1

4.1 Embedding eigenvalues of large boxes into smaller ones

In Section 2, we need to argue that eigenvalues of H_{k+1} localized in C_p produce those of $H_{k,p}$. In order to do that, we review the results in [5, 11]. The following lemma is proved in [5].

Lemma 4.1 *Let $H\phi = E\phi, \phi \in l^2(\mathbf{Z}^d)$ ($H = H_{k+1}$ or $H = H_{k,p}$). Then we can find $L_0(d, \gamma)$ such that for $L \geq L_0$, $\Lambda_L(x(\phi))$ is (γ, E)-singular.*

In the following lemmas, $D_p, C_p, H_{k,p}, H'_{k,p}, H''_{k,p}, S_p$ and T_p are defined in Section 2.

Lemma 4.2 *For any $0 < \gamma' < \gamma$ we can find $k_1 = k_1(\alpha, d, \gamma, \gamma')$ with the following properties.*
(1) Suppose $\omega \in \Omega_k$, $H_{k+1}\phi = E\phi, E \in I, \|\phi\|_{l^2(\Lambda_{k+1})} = 1$ and $X(\phi) \cap C_p \neq \emptyset$ for some $p = 1, 2, \cdots, N_k$. Then if $k \geq k_1$ we have

$$\|(1 - \chi_{D_p})\phi\|_{l^2(\Lambda_{k+1})} \leq e^{-\gamma' \frac{L_{k-1}}{2}},$$

(2) Suppose $\omega \in \Omega_k$, $H_{k+1}\phi = E\phi, E \in I, \|\phi\|_{l^2(\Lambda_{k+1})} = 1$ and $X(\phi) \cap (D_p \setminus C_p) \neq \emptyset$ for some p. Then if $k \geq k_1$, we have

$$\|(1 - \chi_{S_p})\phi\|_{l^2(\Lambda_{k+1})} \leq e^{-\gamma' \frac{L_{k-1}}{2}},$$

(3) Suppose $\omega \in \Omega_k$, $H_{k,p}\phi = E\phi, E \in I, \|\phi\|_{l^2(D_p)} = 1$ and $X(\phi) \cap (D_p \setminus C_p) \neq \emptyset$. Then if $k \geq k_1$, we have

$$\|(1 - \chi_{T_p})\phi\|_{l^2(D_p)} \leq e^{-\gamma' \frac{L_{k-1}}{2}}.$$

Proof. It is sufficient to show (1). Take k_1 large enough with $L_{k_1} \geq L_0(d, \gamma)$. Since $\omega \in \Omega_k$ and since $\Lambda_{k-1}(x(\phi))$ is (γ, E)-singular by Lemma 4.1, $\Lambda_{k-1}(x)$ is (γ, E)-regular for $x \notin D_p$. Therefore, using $|\phi(y)| \leq 1$, we have

$$|\phi(x)| \leq \sum_{\langle y, y' \rangle \in \tilde{\partial}\Lambda_{k-1}(x)} |G_{\Lambda_{k-1}(x)}(E; x, y)||\phi(y')| \leq c_d L_{k-1}^{d-1} e^{-\gamma \frac{L_{k-1}}{2}}.$$

Taking $k_1(\alpha, d, \gamma', \gamma)$ large enough with $L^d_{k_1+1} c_d^2 L^{2(d-1)}_{k_1-1} e^{-\gamma L_{k-1}} \le e^{-\gamma' L_{k-1}}$ gives the result. $\square$

The proof of following lemma is omitted, for it can be shown similarly as Lemma 4.4.

Lemma 4.3 *For any $0 < \gamma' < \gamma$, we can find $k_2 = k_2(\alpha, d, \gamma, \gamma')$ with the following property. Suppose $\omega \in \Omega_k$, $H_{k+1}\phi = E\phi, E \in I$, $\phi \in l^2(\Lambda_{k+1})$ and $X(\phi) \cap C_p \ne \emptyset$ for some p. Then if $k \ge k_2$, we have $N(H_{k,p}, I(E, e^{-\gamma' \frac{L_{k-1}}{2}})) \ge 1$.*

The following lemma is an elementary extension of [11, Lemma 1].

Lemma 4.4 *For any $0 < \gamma' < \gamma$, we can find $k_3(\alpha, d, \gamma, \gamma')$ with the following properties.*

(1) Let $J(\subset I)$ be an interval and suppose $\omega \in \Omega_k$, $H_{k+1}\phi_j = E_j\phi_j$, $j = 1, 2, \cdots, M_p$, $E_j \in J$ and $X(\phi_j) \cap C_p \ne \emptyset$ for some $p = 1, 2, \cdots, N_k$. Then if $k \ge k_3$, we have

$$N(H_{k,p}, J + I(0, \epsilon_{k-1})) \ge M_p, \quad \epsilon_{k-1} = e^{-\gamma' \frac{L_{k-1}}{2}}.$$

(2) Suppose $\omega \in \Omega_k$, $H_{k+1}\phi_j = E_j\phi_j, j = 1, 2, \cdots, M'_p, E_j \in J$ and $X(\phi_j) \cap (D_p \setminus C_p) \ne \emptyset$ for some p. Then if $k \ge k_3$, we have

$$N(H'_{k,p}, J + I(0, \epsilon_{k-1})) \ge M'_p.$$

(3) Suppose $\omega \in \Omega_k, H_{k,p}\phi_j = E_j\phi_j, j = 1, 2, \cdots, M''_p, E_j \in J$ and $X(\phi_j) \cap (D_p \setminus C_p) \ne \emptyset$. Then if $k \ge k_3$, we have

$$N(H''_{k,p}, J + I(0, \epsilon_{k-1})) \ge M''_p.$$

Proof. It is sufficient to show (1). Assume $\|\phi_j\|_{l^2(\Lambda_{k+1})} = 1$ without loss of generality, and set $\psi_j = \chi_{D_p}\phi_j$. Letting $\gamma_m = \frac{\gamma+\gamma'}{2}$, we have by Lemma 4.2,

$$\|\psi_j\|^2_{l^2(D_p)} \ge 1 - e^{-\gamma_m L_{k-1}}, \tag{4.1}$$

$$|\langle \psi_i, \psi_j \rangle_{l^2(D_p)}| \le e^{-\gamma_m L_{k-1}}, \quad i, j = 1, 2, \cdots, M_p,\ i \ne j. \tag{4.2}$$

for $k \ge k_1(\alpha, d, \gamma, \gamma_m)$.

Claim 1
We can find $k'(\alpha, d, \gamma_m)$ such that if $k \geq k'$, $\psi_1, \cdots, \psi_{M_p}$ are linearly independent.

Proof of Claim 1 Otherwise, we can assume

$$\psi_1 + a_2\psi_2 + \cdots + a_{M_p}\psi_{M_p} = 0, \quad |a_j| \leq 1,\ j \geq 2,$$

without loss of generality. Taking the inner-product with ψ_1, we have

$$\langle \psi_1, \psi_1 \rangle_{l^2(D_p)} = -\sum_j a_j \langle \psi_1, \psi_j \rangle_{l^2(D_p)}$$

By $|a_j| \leq 1$ and by (4.1) and (4.2), we have $1 - e^{-\gamma_m L_{k-1}} \leq (M_p - 1)e^{-\gamma_m L_{k-1}}$. Therefore

$$1 \leq M_p e^{-\gamma_m \frac{L_{k-1}}{2}} \leq L_{k+1}^d e^{-\gamma_m \frac{L_{k-1}}{2}}.$$

Taking $k'(\alpha, d, \gamma_m)$ large enough with $L_{k'+1}^d e^{-\gamma_m \frac{L_{k'-1}}{2}} < 1$, we have a contradiction. □

Claim 2

$$\|(H_{k,p} - E_j)\psi_j\|_{l^2(D_p)} \leq \sqrt{2} e^{-\gamma_m L_{k-1}/2}, \quad j = 1, 2, \cdots, M_p.$$

Proof of Claim 2 We decompose

$$H_{k+1} = H_{k,p} + H_{\Lambda_{k+1} \backslash D_p} + \Gamma_{D_p}, \quad \phi_j = \psi_j + \psi_j'.$$

$(H_{k+1} - E_j)\phi_j = 0$ implies $(H_{k,p} - E_j)\psi_j + \Gamma_{D_p}\psi_j' = 0$ so that

$$\|(H_{k,p} - E_j)\psi_j\|_{l^2(D_p)} \leq \|\Gamma_{D_p}\psi_j'\|_{l^2(D_p)} \leq \sqrt{2} e^{-\gamma_m L_{k-1}/2}.$$

Claim 2 is thus proved. □

Let $J' := J + I(0, \epsilon_{k-1})$, let P be the spectral projection corresponding to J' and let $Q = I - P$. Since $\|(H_{k,p} - E_j)Q\psi_j\|_{l^2(D_p)}^2 \geq \epsilon_{k-1}^2 \|Q\psi_j\|_{l^2(D_p)}^2$ by the spectral theorem, we have

$$\|Q\psi_j\|_{l^2(D_p)} \leq \sqrt{2} e^{-(\gamma_m - \gamma')L_{k-1}/2}, \quad j = 1, 2, \cdots, M_p$$

by Claim 2. Let $V :=$ Span $\{\psi_1, \cdots, \psi_{M_p}\}$ and take $\psi \in V, \|\psi\|_{l^2(D_p)} = 1$. Writing $\psi = \sum_j a_j \psi_j$, we have

$$1 = \|\psi\|^2_{l^2(D_p)} = \sum_j |a_j|^2 \|\psi_j\|^2_{l^2(D_p)} + \sum_{i \neq j} a_i \overline{a_j} \langle \psi_i, \psi_j \rangle_{l^2(D_p)}. \tag{4.3}$$

By inequalities (4.1), (4.2) and

$$| \text{ 2nd term of (4.3) } | \leq e^{-\gamma_m L_{k-1}} \sum_{i \neq j} |a_i||a_j| \leq e^{-\gamma_m L_{k-1}} (M_p - 1) \sum_i |a_i|^2,$$

we have $\sum_j |a_j|^2 \leq (1 - M_p e^{-\gamma_m L_{k-1}})^{-1}$ and hence

$$\|Q\psi\|^2_{l^2(D_p)} \leq \sum_j |a_j|^2 \cdot \sum_j \|Q\psi_j\|^2_{l^2(D_p)} \leq \frac{2|\Lambda_{k+1}| e^{-(\gamma_m - \gamma')L_{k-1}}}{1 - |\Lambda_{k+1}| e^{-\gamma_m L_{k-1}}}.$$

Taking $k \geq k_3(\alpha, d, \gamma, \gamma')$ with $\frac{2|\Lambda_{k+1}|e^{-(\gamma_m - \gamma')L_{k-1}}}{1 - |\Lambda_{k+1}|e^{-\gamma_m L_{k-1}}} < \frac{1}{2}$, we have $\|Q\psi\|^2_{l^2(D_p)} < \frac{1}{2}\|\psi\|^2_{l^2(D_p)}$ so that

$$\|P\psi\|^2_{l^2(D_p)} > \frac{1}{2}\|\psi\|^2_{l^2(D_p)}$$

which implies P is injective on V. Therefore dim Ran $P \geq \dim PV = M_p$. $\square$

4.2 Embedding eigenvalues in small boxes into larger ones

In this subsection, we do the converse to what was done in Subsection 4.1 : we argue that an eigenvalues of $H_{k,p}$ localized in C_p produce those of H_{k+1}. Since the proofs are done similarly as in Subsection 4.1, we only state the result.

Lemma 4.5 *For any $0 < \gamma_m < \gamma$, we can find $k_1'(\alpha, d, \gamma, \gamma_m)$ with the following property. Suppose $\omega \in \Omega_k$, $H_{k,p}\phi = E\phi$, $E \in I$, $\|\phi\|_{l^2(D_p)} = 1$ and $X(\phi) \cap C_p \neq \emptyset$. Let $\psi \in l^2(\Lambda_{k+1})$ be the 0-extension of ϕ :*

$$\psi(x) = \begin{cases} \phi(x) & (x \in D_p) \\ 0 & (\textit{otherwise}) \end{cases}$$

Then for $k \geq k_1'$, we have

$$\|(H - E)\psi\|_{l^2(\Lambda_{k+1})} \leq \exp\left(-\gamma_m \frac{L_{k-1}}{2}\right).$$

Lemma 4.6 *For any $0 < \gamma' < \gamma$, we can find $k_2' = k_2'(\alpha, d, \gamma, \gamma')$ with the following property. Suppose $\omega \in \Omega_k$, $H_{k,p}\phi = E\phi$ and $X(\phi) \cap C_p \neq \emptyset$. Then if $k \geq k_2'$, H_{k+1} has an eigenvalue in $J := I(E, \epsilon_{k-1})$, $\epsilon_{k-1} = e^{-\gamma' L_{k-1}/2}$.*

Lemma 4.7 *For any $0 < \gamma' < \gamma$, we can find $k_3' = k_3'(\alpha, d, \gamma, \gamma')$ with the following property. Suppose $\omega \in \Omega_k$, $J (\subset I)$ is an interval, $H_{k,p}\phi_{p,j} = E_{p,j}\phi_{p,j}$, $E_{p,j} \in J$ and $X(\phi_{p,j}) \cap C_p \neq \emptyset$, $j = 1, 2, \cdots, M_p$, $p = 1, 2, \cdots, N_k$. Then if $k \geq k_3'$, we have*

$$N(H_{k+1}, J + I(0, \epsilon_{k-1})) \geq M, \quad M := \sum_{p=1}^{N_k} M_p, \quad \epsilon_{k-1} = e^{-\gamma' L_{k-1}/2}.$$

4.3 A priori estimate

We show a priori estimate for $\mathbf{E}[\xi(\Lambda_{k+1}, E_0)(g)]$ where $\xi(\Lambda_{k+1}, E_0)(J) := \xi_{k+1}(J \times K)$ $J \subset \mathbf{R}$ is defined in Introduction.

Lemma 4.8 *Suppose g is bounded and measurable on $\mathbf{R}$, satisfying*

$$|g(x)| \leq \frac{C_R}{x^2}, \quad |x| \geq R$$

for some $R > 0$ and $C_R > 0$. Let $r := d(E_0, I^c) > 0$. If $r|\Lambda_{k+1}| \geq R$, we have

$$\mathbf{E}[\xi(\Lambda_{k+1}, E_0)(g)] \leq C_W \int_{\{|\lambda| < r|\Lambda_{k+1}|\}} |g(\lambda)| d\lambda + \frac{C_R}{r^2 |\Lambda_{k+1}|}.$$

Proof. We decompose

$$\begin{aligned}
\xi(\Lambda_{k+1}, E_0)(g) &= \sum_j g(|\Lambda_{k+1}|(E_j(\Lambda_{k+1}) - E_0)) \\
&= \sum_{E_j \in I} g(|\Lambda_{k+1}|(E_j(\Lambda_{k+1}) - E_0)) + \sum_{E_j \in I^c} g(|\Lambda_{k+1}|(E_j(\Lambda_{k+1}) - E_0)) \\
&= I + II.
\end{aligned}$$

II is estimated by using the assumption on g.

$$|II| \leq |\Lambda_{k+1}| \cdot \frac{C_R}{r^2 |\Lambda_{k+1}|^2}. \tag{4.4}$$

To estimate I, we note $I = \xi(\Lambda_{k+1}, E_0)(g 1_{\{|\lambda|<r|\Lambda_{k+1}|\}})$. If $g = 1_J$ for some interval $J \subset \{x \in \mathbf{R} : |x| < r|\Lambda_{k+1}|\}$, we have

$$\begin{aligned}\mathbf{E}[\xi(\Lambda_{k+1}, E_0)(g 1_{\{|x|<r|\Lambda_{k+1}|\}})] &= \mathbf{E}[N(H_{k+1}, E_0 + \frac{J}{|\Lambda_{k+1}|})] \\ &\le C_W |J| = C_W \int_{\{|\lambda|<r|\Lambda_{k+1}|\}} |g(\lambda)| d\lambda. \quad (4.5)\end{aligned}$$

A density argument proves (4.5) for g bounded and measurable. Together with (4.4), we arrive at the conclusion. $\square$

5 Appendix 2

In this section, we consider the random measure ξ studied in [10], and examine its natural scaling limit under Assumption A. ξ is defined by

$$\xi(J \times B) := \mathrm{Tr}\ (1_J(x) 1_B(H))$$

for an interval $J \times B$ ($J \subset \mathbf{R}, B \subset \mathbf{R}^d$), and its scaling ξ_L is given by

$$\int f(E, x) d\xi_L := \int f\left(L^d(E - E_0), x/L\right) d\xi, \quad f \in C_c(\mathbf{R}^{d+1}), \quad L > 0$$

which is done in the same spirit of ξ_k. We then have

Theorem 5.1 *Suppose Assumption A (with $p > 8d - 2$) and $E_0 \in I$ is the Lebesgue point of the DS measure dN. Then we can find a convergent subsequence $\{L_k\}_{k=1}^{\infty}$ such that ξ_{L_k} converges in distribution to a infinitely divisible point process ξ on $\mathbf{R}^{d+1}$ with its intensity measure satisfying*

$$\mathbf{E}\xi(dE \times dx) \le n(E_0) dE \times dx.$$

For its proof, we take $l_L = O(L^{\beta})$ for some $0 < \beta < 1$ and consider

$$\begin{aligned} B_p(L) &:= \left\{x \in \mathbf{Z}^d : p_j l_L \le x_j < (p_j + 1) l_L,\ j = 1, \cdots, d\right\}, \quad p \in \mathbf{Z}^d \\ H_{L,p} &:= H|_{B_p(L)}, \quad H_L := \oplus_p H_{L,p} \end{aligned}$$

which is taken under the periodic boundary condition. Let $\tilde{\eta}_{L,p}$ be the random measure defined by

$$\int f(E, x) d\tilde{\eta}_{L,p} := \sum_j \sum_{x \in B_p(L)} f\left(L^d(E_j - E_0), x/L\right) |\psi_j(x)|^2$$

where $\{E_j\}_j, \{\psi_j\}_j$ are eigenvalues and corresponding eigenfunctions of $H_{L,p}$. We then have

Proposition 5.1 *Suppose Assumption A with $p > 8d - 2$. Then for $f \in C_c(\mathbf{R}^{d+1})$,*

$$\mathbf{E}\left[\left|\int f(E,x)d\xi_L - \sum_p \int f(E,x)d\tilde{\eta}_{L,p}\right|\right] \to 0. \tag{5.1}$$

The proof of Proposition 5.1 is done along the following two steps.
Step 1 : We show (5.1) for $f(E,x) = 1_B(x)f_\zeta(E)$ for a box $B \subset \mathbf{Z}^d$ and $\zeta \in \mathbf{C}_+$. f_ζ is defined in Section 3. The proof goes through as [10] except that we use the estimate given by the multiscale analysis instead of the fractional moment boundF

$$\mathbf{P}\left(\sup_{\epsilon>0}|G_\Lambda(E+i\epsilon;x,y)| \le e^{-\frac{\gamma}{8}|x-y|}\right) \ge 1 - C|\Lambda||x-y|^{-p/2} \tag{5.2}$$

for any $E \in I$, any box Λ and any $x, y \in \Lambda$ with $|x-y| \ge C$ for some C, and next argue as in the proof of Lemma 3.1.
Step 2 : We prove a simple estimate

$$\mathbf{E}\left[\left|\int 1_B(x)g(E)d\xi_L\right|\right] \le C_W(1+o(1))|B|\|g\|_1 + \frac{(const.)}{L^d} \tag{5.3}$$

for g bounded and measurable with

$$|g(x)| \le \frac{C_R}{x^2}, \quad |x| \ge R$$

for some $R > 0$ and $C_R > 0$. The estimate (5.3) can be proved similarly as Lemma 3.1 and Lemma 4.8. By a density argument using (5.3), we can show (5.1) for $f(E,x) = 1_B(x)g(E)$ for a box $B \subset \mathbf{Z}^d$ and $g \in C_c(\mathbf{R})$. Then we can further extend (5.1) to arbitrary $f \in C_c(\mathbf{R}^{d+1})$ by some a priori estimates stated below : for any $C > 0$ we can find $L_0(C)$ with

$$\begin{aligned} &(1) \qquad \mathbf{E}\left[\left|\int f(E,x)d\xi_L\right|\right] \le 2n(E_0)\|f\|_1 \\ &(2) \qquad \mathbf{E}\left[\sum_p\left|\int f(E,x)d\tilde{\eta}_{L,p}\right|\right] \le C_W\|f\|_1. \end{aligned} \tag{5.4}$$

for supp $f \subset \{|(E,x)| \le C\}$ and $L \ge L_0(C)$. It is also possible to prove Proposition 5.1 by using the almost analytic extensions.

The facts that the sequence $\{\xi_L\}_L$ is a null-array and relatively compact follow from (5.4), and Proposition 5.1 then proves the infinite divisibility of the limiting random measure ξ. The fact that ξ is a point process and the estimate for its intensity measure $\mathbf{E}\xi(dE \times dx)$ follow similarly as in [10], completing the proof of Theorem 5.1.

We end this section with some remarks.

Remark 5.1 *Let $B(\subset \mathbf{Z}^d)$ be a finite box and let $H_{LB} := H|_{LB}$ be a restriction of H on LB with some boundary condition with $\{E_j\}, \{\psi_j\}$ its eigenvalues and corresponding eigenfunctions. Define a random measure $\xi_{L,B}$ by*

$$\int f(E,x) d\xi_{L,B} := \sum_j \sum_x f(L^d(E_j - E_0), x/L)|\psi_j(x)|^2.$$

Then for $f \in C_c(\mathbf{R} \times B)$, the proof of Proposition 5.1 tells us that

$$\mathbf{E}\left[\left| \int f(E,x) d\xi_{L,B} - \sum_p \int f(E,x) d\tilde{\eta}_{L,p} \right| \right] = o(1).$$

Together with Proposition 5.1, we have

$$\mathbf{E}\left[\left| \int f(E,x) d\xi_{L,B} - \int f(E,x) d\xi_L \right| \right] = o(1), \quad f \in C_c(\mathbf{R} \times B)$$

which shows $\xi_L - \xi_{L,B} \to 0$ vaguely a.s. Therefore the eigenvalues and the eigenfunctions on H_{LB} and those of H localized in LB has the same behavior in this sense.

Remark 5.2 *If we would try to prove that the limiting point process is unique without Assumption B, it is sufficient to find a random variable ξ_f with*

$$\xi_L(f) \xrightarrow{d} \xi_f$$

for $f(E,x) = 1_B(x) f_\zeta(E), \zeta \in \mathbf{C}_+$ ($B(\subset \mathbf{Z}^d)$ is any finite box). Here we used (5.3) and [9, Lemma 5.1]. This is equivalent to proving that the $L \to \infty$ limit of the function

$$\begin{aligned} L_{\xi_L}(f) &= \mathbf{E}[\exp(-\xi_L(1_B(x) f_\zeta(E)))] \\ &= \mathbf{E}\left[\exp\left(-L^{-d} \sum_{x \in LB} \Im G(E_0 + L^{-d}\zeta; x, x) \right) \right] \end{aligned}$$

is the Laplace transform of a random variable. Then one would have to handle various limits including the non-tangential limit of Green's function, while, under Assumption B, $L_{\xi_L}(f)$ converges to

$$\mathbf{E}[\exp\left(-\zeta_{P,\mathbf{R}^{d+1}}1_B(x)f_\zeta(E)\right)] = \exp\left(-n(E_0)|B|\int_{\mathbf{R}} dE(1-e^{-f_\zeta(E)})\right).$$

Remark 5.3 *It is known that H has the semi-uniformly localized eigenfunction(SULE) : the eigenfunctions $\{\varphi_n\}$ of H satisfy*

$$|\varphi_n(x)| \le C_{\omega,\epsilon,\gamma_0} e^{\gamma_0|x_n|^\epsilon} e^{-\gamma_0|x-x_n|}, \quad n = 1, 2, \cdots,$$

almost surely, where $x_n \in X(\varphi_n)$, $\gamma_0 > 0$, $\epsilon > 0$, and this estimate can not be improved in general [4]. However, the factor $e^{\gamma_0|x_n|^\epsilon}$ does not play a serious role to prove Proposition 5.1 ; In fact, since only the finite volume is involved in our situation, we can prove

$$|\varphi_n(x)| \le C_\omega e^{-\gamma_0|x-x_n|}, \quad n = 1, 2, \cdots,$$

for eigenfunctions whose localization centers lie in a finite box B.

6 Appendix 3

In this section, we assume both Assumption A and B and present another proof of Proposition 2.1. We use the notation in Section 2 and for simplicity, let

$$J'_{k+1} := J_{k+1} + I(0, \epsilon_{k-1}).$$

As was done in the proof of Proposition 2.1, we take $k \gg 1$ so that Lemma 4.3, 4.4, 4.6, 4.7 are applicable. By Step 1 and Step 2 in the proof of Proposition 2.1, it suffices to show the following equation for $f \in C_c(\mathbf{R} \times K)$.

$$\sum_p \mathbf{E}\left[|\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)|; \Omega_k\right] = o(1). \tag{6.1}$$

We first decompose the LHS of (6.1) as

$$\begin{aligned}\text{LHS of (6.1)} &= \sum_p \mathbf{E}\left[|\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1\}\right] \\ &\quad + \sum_p \mathbf{E}\left[|\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) \ge 2\}\right] \\ &=: A + B.\end{aligned}$$

We note that, since

$$N(H_{k+1}, J_{k+1}, C_p) \le N(H_{k,p}, J'_{k+1}), \quad \omega \in \Omega_k \tag{6.2}$$

by Lemma 4.4, $|\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)| = 0$ if $N(H_{k,p}, J'_{k+1}) = 0$.

Claim 1 : $B = o(1)$.

Proof of Claim 1 We write $B = \sum_p B_p$. By (6.2) and by Assumption B, we have

$$\begin{aligned}
B_p &\le \|f\|_\infty \mathbf{E}\left[N(H_{k+1}, J_{k+1}, C_p) + N(H_{k,p}, J'_{k+1}); \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) \ge 2\}\right] \\
&\le 2\|f\|_\infty \mathbf{E}\left[N(H_{k,p}, J'_{k+1}); \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) \ge 2\}\right] \\
&= 2\|f\|_\infty \sum_{j\ge 2} j(j-1)\mathbf{P}\left(N(H_{k,p}, J'_{k+1}) = j\right) \\
&\le 2\|f\|_\infty C_M \left(\frac{2a}{|\Lambda_{k+1}|} + 2\epsilon_{k-1}\right)^2 \cdot |\Lambda_k|^2
\end{aligned}$$

which shows $B \le (const.)\frac{|\Lambda_k|}{|\Lambda_{k+1}|}$ and thus proves Claim 1. □

To estimate A, we further decompose $A = A_1 + A_2$ with

$$\begin{aligned}
A_1 &= \sum_p \mathbf{E}\left[|\eta_{k+1,p}(f) - \xi^{(1)}_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1,\ N(H_{k+1}, J_{k+1}, C_p) = 1\}\right] \\
A_2 &= \sum_p \mathbf{E}\left[|\eta_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1,\ N(H_{k+1}, J_{k+1}, C_p) = 0\}\right].
\end{aligned}$$

Claim 2 : $A_2 = o(1)$.

Proof of Claim 2 While it is possible to prove Claim 2 by the argument used in Step 3 in the proof of Proposition 2.1, we take an another route here. The argument in the proof of Claim 1 gives

$$\begin{aligned}
&\mathbf{E}[N(H_{k+1}, J_{k+1})] \\
&= \sum_p \mathbf{E}\left[N(H_{k+1}, J_{k+1}, C_p); \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1\}\right] + o(1).
\end{aligned} \tag{6.3}$$

On the other hand, by Lemma 6.1 below and by the argument in the proof of Claim 1 again,

$$\begin{aligned}\mathbf{E}[N(H_{k+1}, J_{k+1})] &= \sum_p \mathbf{E}[N(H_{k,p}, J'_{k+1})] + o(1) \\ &= \sum_p \mathbf{E}\left[N(H_{k,p}, J'_{k+1}); \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1\}\right] + o(1). \quad (6.4)\end{aligned}$$

By (6.2), (6.3) and (6.4), we have

$$0 \le \sum_p \mathbf{E}\left[N(H_{k,p}, J'_{k+1}) - N(H_{k+1}, J_{k+1}, C_p); \Omega_k \cap \{N(H_{k,p}, J'_{k+1}) = 1\}\right] = o(1). \quad (6.5)$$

Since we have

$$|\eta_{k+1,p}(f)| \le \|f\|_\infty \left(N(H_{k,p}, J'_{k+1}) - N(H_{k+1}, J_{k+1}, C_p)\right)$$

on the event in which A_2 is computed, (6.5) implies $A_2 = o(1)$ and thus proves Claim 2. □

By Lemma 4.3, for an eigenvalue E_p of H_{k+1} in J_{k+1} localized in C_p, we can find an eigenvalue $F_p \in I(E_p, \epsilon_{k-1})$ of $H_{k,p}$ in J'_{k+1}. Since $N(H_{k+1}, J_{k+1}, C_p) = 1$ and $N(H_{k,p}, J'_{k+1}) = 1$ on the events in which A_1 is computed, there are no other eigenvalues of $H_{k,p}$ in J'_{k+1}. Then $A_1 = o(1)$ follows from the argument in Step 3 in the proof of Proposition 2.1.

It remains to show the following lemma.

Lemma 6.1 *If $p > 12d$ in Assumption A(1), we have*

$$\mathbf{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbf{E}[N(H_{k,p}, J'_{k+1})] + o(1).$$

Proof. Since

$$\sum_p \mathbf{E}[N(H_{k,p}, J'_{k+1} \setminus J_{k+1})] \le C_W |\Lambda_k| 2\epsilon_{k-1} \frac{|\Lambda_{k+1}|}{|\Lambda_k|} = o(1)$$

by Wegner's estimate, it suffices to show

$$\mathbf{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbf{E}[N(H_{k,p}, J_{k+1})] + o(1).$$

Let $\tilde{f}(E,x) = f(E)$ for $f \in C_c(\mathbf{R})$. By Lemma 2.1, it is further reduced to

$$\mathbf{E}\left[\left|\xi_{k+1}(\tilde{f}) - \sum_p \eta_{k+1,p}(\tilde{f})\right|\right] = o(1).$$

By the argument in [14, Step 1], it is sufficient to take $f = f_\zeta$, $\zeta \in \mathbf{C}_+$ in which case the proof can be done by using the argument in [14, Step 3] and (5.2). $\square$

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